

Generalized Hirota Equations in Models of 2D Quantum Gravity

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We derive a set of bilinear functional equations of Hirota type for the partition functions of the $sl(2)$ related integrable statistical models defined on a random lattice. These equations are obtained as deformations of the Hirota equations for the KP integrable hierarchy, which are satisfied by the partition function of the ensemble of planar graphs.

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1. Introduction

The formulation of 2D quantum gravity in terms of random matrix variables opened the possibility to apply powerful nonperturbative techniques as the method of orthogonal polynomials [1] and led to the discovery of unexpected integrable structures associated with its different scaling regimes. Originally, a structure related to the KdV hierarchy of soliton equations was found in the continuum limit of pure gravity [2]. Consequently, the partition functions of the A series of models of matter coupled to 2D gravity were identified as τ -functions of higher reductions of the KP integrable hierarchy [3]. Similar statement concerning the D series was made in [4]. The reductions of the KP hierarchy reflect the flow structure of the theory of interacting gravity and matter at distances much smaller than the correlation length of the matter fields. On the other hand, there are interesting phenomena as the topology changing interactions and, in general, all processes involving world manifolds with negative curvature, which become important at distances much larger than the correlation length of the matter fields. At extra large distances the fluctuations of the matter fields are not important and the flow structure of the theory becomes the one of pure gravity. Therefore, to study the infrared phenomena, we need a complementary description in which the matter is considered as a perturbation of pure gravity. The simplest realization of such a description is given by the Kazakov's multicritical points of the one-matrix model [5], where “pure” gravity theories defined as specially weighted lattices were shown to be equivalent to theories of gravity coupled to (nonunitary) matter fields.

In this letter we propose a systematic construction of the theories with matter fields as deformations of the integrable structure of pure gravity. For this purpose we will exploit the microscopic realization of the matter degrees of freedom given by the $sl(2)$ -related statistical models: the Ising [6] and the $O(n)$ [7] models, the SOS and RSOS models and their ADE and $\hat{A}\hat{D}\hat{E}$ generalizations [8]-[9]. Each of these models can be reformulated as a theory of one or several random matrices with interaction of the form $\text{tr} \ln(M_a \otimes 1 + 1 \otimes M_b)$. Such an interaction can be introduced by means of a differential operator of second order \mathbf{H} acting on the coupling constants; the partition function of the model is obtained by acting with the operator $e^{\mathbf{H}}$ on the partition function of one or several decoupled one-matrix integrals. The Virasoro constraints $L_n^a = 0$ for each of the one-matrix integrals transform to linear differential constraints $e^{\mathbf{H}} L_n^a e^{-\mathbf{H}} = 0$ for the interacting theory. These constraints are equivalent to the loop equations, which have been already derived by other means [8] and therefore will not be discussed here. The new point is that the Hirota bilinear

equations that hold for each of the one-matrix integrals become, upon replacing the vertex operators as $\mathbf{V}_\pm(z)^a \rightarrow e^{\mathbf{H}} \mathbf{V}_\pm^a(z) e^{-\mathbf{H}}$, bilinear functional equations for the interacting theory. In this way the integrable structure associated with pure gravity is deformed but not destroyed by the matter fields.

We start with deriving the Hirota equations for the one-matrix integral using the formalism of orthogonal polynomials. Then we show how these equations generalize for the matrix integrals describing theories of matter fields coupled to gravity. We restrict ourselves to the microscopic description of the theory, but our construction survives without substantial changes in the continuum limit where the loop equations and the bilinear equations are obtained as deformations of the Virasoro constraints and the Hirota equations in the KdV hierarchy. Their perturbative solution is given by the loop-space Feynman rules obtained in [10].

2. Orthogonal polynomials and Hirota equations in the one-matrix model

The partition function of the ensemble of all two-dimensional random lattices is given by the hermitian $N \times N$ matrix integral

$$Z_N[t] \sim \int dM \exp \left(\text{tr} \sum_{n=0}^{\infty} t_n M^n \right) \quad (2.1)$$

or, in terms of the eigenvalues λ_i , $i = 1, \dots, N$,

$$Z_N[t] = \int \prod_{i=1}^N d\lambda_i \exp \left(\sum_{n=0}^{\infty} t_n \lambda_i^n \right) \prod_{i < j} (\lambda_i - \lambda_j)^2. \quad (2.2)$$

For the time being we restrict the integration in λ 's to a finite interval $[\lambda_L, \lambda_R]$ on the real axis, so that the measure

$$d\mu_t(\lambda) = d\lambda \exp \left(\sum_{n=0}^{\infty} t_n \lambda^n \right) \quad (2.3)$$

is integrable for any choice of the coupling constants t_n .

It is known [11] that the partition function (2.1) is a τ -function of the KP hierarchy of soliton equations. The global form of this hierarchy is given by the Hirota's bilinear equations [12] (for a review on the theory of the τ -functions see, for example, [13].) Below we will derive the Hirota equations using the formalism of orthogonal polynomials.

Introduce for each N ($N = 0, 1, 2, \dots$) the polynomial

$$\begin{aligned} P_{N,t}(\lambda) &= \langle \det(\lambda - M) \rangle_{N,t} \\ &= \frac{1}{Z_N} \int \prod_{i=1}^N d\mu_t(\lambda_i) (\lambda - \lambda_i) \prod_{i < j} (\lambda_i - \lambda_j)^2. \end{aligned} \quad (2.4)$$

It is easy to prove [1] that the polynomials (2.4) are orthogonal with respect to the measure $d\mu_t(\lambda)$. Indeed,

$$\begin{aligned} & Z_N \int d\mu_t(\lambda_{N+1}) P_{N,t}(\lambda_{N+1}) \lambda_{N+1}^k \\ &= \frac{1}{N+1} \int \prod_{i=1}^{N+1} d\mu(\lambda_i) \Delta_{N+1}(\lambda_1, \dots, \lambda_{N+1}) \Delta_N(\lambda_1, \dots, \lambda_N) \lambda_{N+1}^k \\ &= \frac{1}{N+1} \int \prod_{i=1}^{N+1} d\mu_t(\lambda_i) \Delta_{N+1}(\lambda_1, \dots, \lambda_{N+1}) \\ & \quad \sum_{s=1}^{N+1} (-)^{N+1-s} \lambda_s^k \Delta_{N+1}(\lambda_1, \dots, \hat{\lambda}_s, \dots, \lambda_{N+1}). \end{aligned} \quad (2.5)$$

The sum in the integrand is the expansion of the determinant

$$\begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} & \lambda_1^k \\ 1 & \lambda_2 & \dots & \lambda_2^{N-1} & \lambda_2^k \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \lambda_{N+1} & \dots & \lambda_{N+1}^{N-1} & \lambda_{N+1}^k \end{vmatrix}$$

with respect to its last column. It vanishes for $k = 1, \dots, N-1$, which proves the statement.

For $n = N$, one finds

$$Z_N \int d\mu_t(\lambda) P_{N,t}(\lambda) \lambda^N = Z_{N+1}/(N+1). \quad (2.6)$$

Hence

$$\int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda) P_{N,t}(\lambda) P_{k,t}(\lambda) = \delta_{N,k} \frac{Z_{N+1}[t]}{(N+1)Z_N[t]}. \quad (2.7)$$

The orthogonality relations (2.7) can be written in the form of a contour integral, namely,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} dz \left\langle \det(z - M) \right\rangle_{N,t} \left\langle \frac{1}{\det(z - M)} \right\rangle_{[k+1,t]} = \delta_{N,k} \quad (2.8)$$

where the integration contour \mathcal{C} encloses the point $z = 0$ and the interval $[\lambda_L, \lambda_R]$. Indeed, the residue of each of the n poles is equal to the left hand side of (2.7) multiplied by $Z_k[t]/Z_{k+1}[t]$.

A set of more powerful identities follow from the fact that the polynomial $P_{N,t}(z)$ is orthogonal to *any* polynomial of degree less than N and in particular to the polynomials $P_{k,t'}[\lambda], k = 1, 2, \dots, N-1$ where $t' = \{t'_n, n = 1, 2, \dots\}$ is another set of coupling constants. Written in the form of contour integrals, these orthogonality relations state, for $N' \leq N$,

$$\oint_{\mathcal{C}} dz e^{\sum_{n=1}^{\infty} (t_n - t'_n) z^n} \left\langle \det(z - M) \right\rangle_{N,t} \left\langle \frac{1}{\det(z - M)} \right\rangle_{N',t'} = 0. \quad (2.9)$$

The Hirota equations for the KP hierarchy are obtained from eq. (2.9) after expressing the mean values of $\langle \det^{\pm 1} \rangle$ in terms of the vertex operators

$$\mathbf{V}_{\pm}(z) = \exp \left(\pm \sum_{n=0}^{\infty} t_n z^n \right) \exp \left(\mp \ln \frac{1}{z} \frac{\partial}{\partial t_0} \mp \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n} \right). \quad (2.10)$$

It follows from the definition (2.2) that

$$\mathbf{V}_{\pm}(z) \cdot Z_N[t] = e^{\pm \sum_{n=0}^{\infty} t_n z^n} \left\langle \det(z - M)^{\pm 1} \right\rangle_{N,t} Z_N[t] \quad (2.11)$$

and eq. (2.9) is therefore equivalent to

$$\oint_{\mathcal{C}} dz \left(\mathbf{V}_+(z) \cdot Z_N[t] \right) \left(\mathbf{V}_-(z) \cdot Z_{N'}[t'] \right) = 0 \quad (N' \leq N), \quad (2.12)$$

which is one of the forms of the Hirota equation for KP [13].

After a change of variables

$$x_n = \frac{t_n + t'_n}{2}, \quad y_n = \frac{t_n - t'_n}{2}, \quad (2.13)$$

the Hirota equations (2.12) take its canonical form

$$\text{Res}_{z=0} z^{N-N'} e^{\sum_{n=1}^{\infty} 2y_n z^n} e^{-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial y_n}} Z_N[x+y] Z_{N'}[x-y] = 0 \quad (2.14)$$

where $N' \leq N$. The differential equations of the KP hierarchy are obtained by expanding (2.14) in y_n . For example, for $N' = N$, the coefficient in front of y_1^3 is

$$\left(\frac{\partial^4}{\partial y_1^4} + 3 \frac{\partial^2}{\partial y_2^2} - 4 \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_3} \right) Z_N[t+y] Z_N[t-y] \Big|_{y=0} = 0 \quad (2.15)$$

and one finds for the “free energy” $u[t] = 2 \frac{\partial^2}{\partial t_1^2} \log Z_N$ the KP equation

$$3 \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1} \left[-4 \frac{\partial u}{\partial t_3} + 6u \frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3} \right] = 0. \quad (2.16)$$

The lowest equation in the case $N' = N - 1$ (modified KP) is the so called Miura transformation relating the functions $u[t]$ and $v[t] = \log(Z_{N+1}[t]/Z_N[t])$:

$$u = \partial_2 v - \partial_1^2 v - (\partial_1 v)^2. \quad (2.17)$$

3. Bilinear functional equations in the $O(n)$ model

The partition function of the $O(n)$ matrix model¹ is defined by the $N \times N$ matrix integral [7]

$$Z_N^{O(n)}[t] \sim \int dM \exp \left[\sum_{n=0}^{\infty} t_n \operatorname{tr} M^n + \frac{n}{2} \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! m!} \operatorname{tr} M^n \operatorname{tr} M^m \right]. \quad (3.1)$$

(the sum over n and m runs the set of nonnegative integers) and describes the ensemble of nonintersecting loops on a random graph. A loop of length n is weighted by a factor nT^{-n} where T is the temperature of the loop gas. For $n \in [-2, 2]$, the model exhibits critical behavior with spectrum of the central charge $C = 1 - 6(g-1)^2/g$, $g = \frac{1}{\pi} \arccos(-n/2)$; the critical behavior for $|n| > 2$ is this of a branched polymer [14]. The matrix integral (3.1) can be again interpreted as the partition function of a Coulomb gas with more complicated but still pairwise interaction between particles:

$$Z_N^{O(n)}[t] = T^{-\frac{1}{2}nN^2} \int \prod_{i=1}^N \frac{d\lambda_i}{(T - 2\lambda_i)^{n/2}} e^{\sum_n t_n \lambda_i^n} \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{(T - \lambda_i - \lambda_j)^n}. \quad (3.2)$$

The interval of integration $[\lambda_L, \lambda_R]$ should be such that $\lambda_R \leq T/2$. With the last restriction the denominator in the integrand never vanishes. The choice of the integration interval does not influence the quasiclassical expansion and hence the geometrical interpretation in terms of a gas of loops on the random planar graph. The saddle point spectral density is automatically supported by an interval on the half line $[-\infty, \frac{T}{2}]$ [7]. The most natural choice for the eigenvalue interval is therefore $\lambda_L \rightarrow -\infty, \lambda_R \rightarrow T/2$, with the conditions $\frac{d\mu(\lambda)}{d\lambda}|_{-\infty} = \frac{d\mu(\lambda)}{d\lambda}|_{\frac{T}{2}} = 0$.

The $O(n)$ model reduces to the hermitian matrix model in the limit $n \rightarrow 0$ and/or $T \rightarrow \infty$, and can be considered as a deformation of the latter in the following sense. Let us define the differential operator

$$\mathbf{H} = \frac{n}{2} \left[-\ln \frac{1}{T} \frac{\partial^2}{\partial t_0^2} + \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! m!} \frac{\partial}{\partial t_n} \frac{\partial}{\partial t_m} \right] \quad (3.3)$$

¹ We use a Roman letter for the parameter $n \in [-2, 2]$ to avoid confusion with the summation index n running the set of natural numbers.

acting on the coupling constants. It is easy to see that the partition function (3.2) is obtained from the partition function of the one-matrix model by acting with the operator $e^{\mathbf{H}}$:

$$Z_N^{O(n)}[t] = e^{\mathbf{H}} \cdot Z_N[t]. \quad (3.4)$$

This simple observation will be of crucial importance for our further consideration. It means that the integrable structure of the one-matrix model survives in some form in the $O(n)$ model.

The Hirota equations (2.12) provide, due to the relation (3.4), a set of bilinear equations for the partition function of the $O(n)$ model

$$\oint_{C_-} dz \left(\tilde{\mathbf{V}}_+(z) \cdot Z_N^{O(n)}[t] \right) \left(\tilde{\mathbf{V}}_-(z) \cdot Z_{N'}^{O(n)}[t'] \right) = 0 \quad (N' \leq N) \quad (3.5)$$

where the integration contour C_- encloses the interval $[\lambda_L, \lambda_R]$ and leaves outside point z and the interval $[T - \lambda_R, T - \lambda_L]$, and $\tilde{\mathbf{V}}_{\pm}(z) = e^{\mathbf{H}} \mathbf{V}_{\pm}(z) e^{-\mathbf{H}}$ are the transformed vertex operators whose explicit form is

$$\begin{aligned} \tilde{\mathbf{V}}_{\pm} = & e^{\pm \sum_{n=1}^{\infty} t_n z^n} \exp \left(\mp \left[\ln \frac{1}{z} \frac{\partial}{\partial t_0} + \sum_{n=1}^{\infty} z^{-n} \frac{\partial}{\partial t_n} \right] \right) \\ & (T - 2z)^{-n/2} \exp \left(\pm n \left[\ln \frac{1}{(T - z)} \frac{\partial}{\partial t_0} + \sum_{n=1}^{\infty} \frac{(T - z)^{-n}}{n} \frac{\partial}{\partial t_n} \right] \right). \end{aligned} \quad (3.6)$$

After being expanded in $t_n - t'_n$, eq. (3.5) generates a hierarchy of differential equations, each of them involving derivatives with respect to an infinite number of “times” t_n .

The functional relations (3.5) are equivalent to the bilinear relations

$$\begin{aligned} & \oint_{C_-} \frac{dz}{(T - 2z)^n} \exp \left(\sum_{n=1}^{\infty} (t_n - t'_n) z^n \right) \\ & \left\langle \frac{\det(z - M)}{\det(T - z - M)^n} \right\rangle_{N,t} \left\langle \frac{\det(T - z - M)^n}{\det(z - M)} \right\rangle_{[N',t']} = 0 \quad (N' \leq N) \end{aligned} \quad (3.7)$$

where $\langle \rangle_{N,t}$ denotes the average corresponding to the partition function (3.2). These relations can be also proved directly by exchanging the order of the integration in the λ 's and the contour integration in z .

4. Bilinear functional equations in the ADE and $\hat{A}\hat{D}\hat{E}$ models

The ADE and $\hat{A}\hat{D}\hat{E}$ matrix models give a nonperturbative microscopic realization of the rational string theories with $C \leq 1$. Each one of these models is associated with a rank r classical simply laced Lie algebra (that is, of type $A_r, D_r, E_{6,7,8}$) or its affine extension, and represents a system of r coupled random matrices. The matrices M_a of size $N_a \times N_a$ ($a = 1, 2, \dots, r$) are associated with the nodes of the Dynkin diagram of the simply laced Lie algebra, the latter being defined by its adjacency matrix G with elements

$$G^{ab} = \begin{cases} 1 & \text{if the two nodes are the extremities of a link } \langle ab \rangle \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

The partition function $Z_N^G[\vec{t}]$ depends on r sets of coupling constants $\vec{t} = \{t_n^a \mid a = 1, \dots, r; n = 1, 2, \dots\}$. The interaction is of nearest-neighbor type and the measure is a product of factors associated with the nodes a and the links $\langle ab \rangle$ of the Dynkin diagram

$$Z_N^G[\vec{t}] \sim \int \prod_{a=1}^r dM_a \exp \left(\sum_{a=1}^r \sum_{n=1}^{\infty} t_n^a \text{tr} M_a^n + \frac{1}{2} \sum_{a,b} G^{ab} \sum_{m,n=1}^{\infty} \frac{T^{-m-n}}{m+n} \frac{(m+n)!}{m! n!} \text{tr} M_a^m \text{tr} M_b^n \right). \quad (4.2)$$

The target space of the A_r model is an open chain of r points and its critical regimes describe theories of $C = 1 - 6 \frac{(g-1)^2}{g}$ matter coupled to gravity, $g = 1 \pm \frac{1}{r+1} + 2m$, $m \in \mathbb{Z}_+$. The target space of the \hat{A}_{r-1} model is a circle with $r+1$ points and its continuum limit describes a compactified gaussian field coupled to gravity. The radius of compactification is r in a scale where the self-dual radius is $r = 2$. In this sense the $O(2)$ model can be referred to as the \hat{A}_0 model of the \hat{A} series.

Again, the only nontrivial integration is with respect to the eigenvalues λ_{ai} ($i = 1, \dots, N_a$) of the matrices M_a :

$$Z_N^G[\vec{t}] = \prod_{a=1}^r \prod_{i=1}^{N_a} d\lambda_{ai} e^{\sum_n t_n^a \lambda_{ai}^n} \frac{\prod_a \prod_{i < j} (\lambda_{ai} - \lambda_{aj})^2}{\prod_{\langle ab \rangle} \prod_{i,j} (T - \lambda_{ai} - \lambda_{bj})}. \quad (4.3)$$

The domain of integration is assumed to be a compact interval $[\lambda_L, \lambda_R]$ with $\lambda_R \leq T/2$.

The partition function (4.3) can be obtained by acting on the product of r one-matrix partition functions $Z_{N_a}[t^a]$, $a = 1, \dots, r$, with the exponent of the second-order differential operator

$$\mathbf{H} = \frac{1}{2} \sum_{a,b} G^{ab} \left[\ln T^{-1} \frac{\partial}{\partial t_0^a} \frac{\partial}{\partial t_0^b} + \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! m!} \frac{\partial}{\partial t_n^a} \frac{\partial}{\partial t_m^b} \right], \quad (4.4)$$

namely,

$$Z_{\vec{N}}^G[\vec{t}] = e^{\mathbf{H}} \cdot \prod_{a=1}^r Z_{N_a}[t^a]. \quad (4.5)$$

Each of the one-matrix partition functions on the right hand side of (4.5) satisfies the Hirota equations (2.12). As a consequence, the left hand side satisfies a set of r functional equations associated with the nodes $a = 1, \dots, r$ of the Dynkin diagram. Introducing the transformed vertex operators

$$\tilde{\mathbf{V}}_{\pm}^a(z) = e^{\mathbf{H}} \mathbf{V}_{\pm}^a(z) e^{-\mathbf{H}} \quad (4.6)$$

we find, for each node $a = 1, \dots, r$,

$$\oint_{\mathcal{C}_-} dz \left(\tilde{\mathbf{V}}_+^a(z) \cdot Z_{\vec{N}}^G[\vec{t}] \right) \left(\tilde{\mathbf{V}}_-^a(z) \cdot Z_{\vec{N}'}^G[\vec{t}'] \right) = 0 \quad (N'_a \leq N_a) \quad (4.7)$$

where the integration contour \mathcal{C}_- in (4.7) encloses the interval $[\lambda_L, \lambda_R]$ but leaves outside the interval $[T - \lambda_R, T - \lambda_L]$.

Inserting (4.4) in the definition (4.6) we find the explicit form of the dressed vertex operators

$$\begin{aligned} \tilde{\mathbf{V}}_{\pm}^a(z) = & \prod_b (T - 2z)^{-\frac{1}{2}G^{ab}} e^{\pm \sum_{n=0}^{\infty} t_n^a z^n} \exp \left(\mp \left[\ln z^{-1} \frac{\partial}{\partial t_0^a} + \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n^a} \right] \right) \\ & \exp \left(\pm \sum_b G^{ab} \left[\ln(T - z)^{-1} \frac{\partial}{\partial t_0^b} + \sum_{n=1}^{\infty} \frac{(T - z)^{-n}}{n} \frac{\partial}{\partial t_n^b} \right] \right) \end{aligned} \quad (4.8)$$

and write the generalized Hirota equations (4.7) in terms of correlation functions of determinants:

$$\begin{aligned} & \oint_{\mathcal{C}} dz \frac{e^{\sum_{n=1}^{\infty} [t_n^a - t_n'^a] z^n}}{\prod_b (T - 2z)^{G^{ab}}} \left\langle \frac{\det(z - M_a)}{\prod_b \det(T - z - M_b)^{G^{ab}}} \right\rangle_{\vec{N}, \vec{t}} \\ & \left\langle \frac{\prod_b \det(T - z - M_b)^{G^{ab}}}{\det(z - M_a)} \right\rangle_{[\vec{N}', \vec{t}']} = 0 \quad (N'_a \leq N_a). \end{aligned} \quad (4.9)$$

Here $\langle \cdot \rangle_{\vec{N}, \vec{V}}$ denotes the average in the ensemble described by the partition function (4.3). It is possible to prove eq. (4.9) directly by performing the contour integration. It is essential for the proof that the interval $[T - \lambda_R, T - \lambda_L]$ is outside the contour \mathcal{C} .

5. Concluding remarks

We proposed an alternative nonperturbative formulation of 2D gravity suited for investigating the infrared properties of the theory. We used the fact that in the microscopic realization based on the $sl(2)$ integrable statistical models, the matter can be introduced as deformation of pure gravity. As a consequence, the integrable structure of pure gravity is not destroyed and manifests itself in the form of bilinear functional equations of Hirota type. The main difference between the loop equations and the bilinear equations is that the first relate correlation functions boson-like quantities (traces) while the second relate correlation functions of fermion-like quantities (determinants).

The continuum version of these equations will be considered elsewhere. They have the same form as the equations in the microscopic theory, the only difference being that the vertex operators in the continuum theory are expanded in the half-integer powers of the shifted and rescaled variable z . The corresponding coupling constants are related to the coupling constants associated with scaling operators as the coefficients in the Laurent expansions of the same function at two different points: near the edge of the eigenvalue interval and near infinity.

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